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Uniform exponential dichotomy of stochastic cocycles

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Abstract

The aim of this paper is to give a generalization of the well-known theorem of Perron for uniform exponential dichotomy in mean square for stochastic cocycles in Hilbert spaces.

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Keywords: Stochastic semiflow; Stochastic cocycles; Uniform exponential dichotomy

1. Introduction

In the qualitative theory of evolution equations, exponential dichotomy is one of the most important asymptotic properties. In the last few years, it has been treated from various perspectives.

The notion of exponential dichotomy for linear differential equations was introduced in a paper [11] by Perron in 1930, which was concerned with the problem of conditional stability of a deterministic differential equation $x'(t) = A(t)x(t)$ and its connection with the existence of bounded solutions of the perturbed equation $x'(t) = A(t)x(t) + \alpha(t)$.

The nonuniform asymptotic behavior of linear evolutionary processes in terms of Lyapunov equations is presents in [5].

In [1,2], Arnold present the random dynamical system generated by random differential equations and stochastic differential equations.

A sufficient condition for stochastic Ito systems to be exponentially dichotomous is proves in [3].

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The problem of the existence of stochastic semiflows for semilinear stochastic evolution equations is a non-trivial one, mainly due to the well-established fact that finite-dimensional methods for constructing (even continuous) stochastic flow break down in the infinite-dimensional setting of semilinear stochastic evolution equations (see [7,8,14]). For linear stochastic evolution equations with finite-dimensional noise, a stochastic semiflow (i.e. a random evolution operator) was obtained in [4].

In [10], the existence of perfect differentiable cocycles generated by mild solutions of a large class of semilinear stochastic evolution equations (sees) and stochastic partial differential equations (spdes) is proved.

In [15,16] the author presents the uniform exponential stability and instability for stochastic cocycles.

In this paper we consider the general case of stochastic cocycles over stochastic semiflows.

The main objective is to give a generalization of the classical Perron result for uniform exponential dichotomy in mean square of stochastic cocycles in Hilbert spaces.

2. Stochastic cocycles

Let $\left(\Omega, F, \{F_t\}_{t\geq 0}, \mathbf{P}\right)$ be a standard filtered probability space, i.e. $(\Omega, F, \mathbf{P})$ is a probability space, $\{F_t\}_{t\geq 0}$ is an increasing family of σ-algebras, F_0 contains all $\mathbf{P}$-null sets of F and

$$F_t = \bigcap_{s\geq t} F_s, \quad \text{for every } t \geq 0.$$

Definition 2.1. A *stochastic semiflow on* Ω is a measurable random field $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$ satisfying the following properties:

(f1) $\varphi(0, \omega) = \omega$,

(f2) $\varphi(t + s, \omega) = \varphi(t, \varphi(s, \omega))$

for all $(t, s, \omega) \in R_+^2 \times \Omega$.

Example 2.1. Let X be a real separable Hilbert space and let Ω be the space of all continuous paths $\omega : \mathbf{R}_+ \to X$, such that $\omega(0) = 0$ with the compact open topology.

Let F_t, for $t \geq 0$, be the σ-algebra generated by the set $\{\omega \to \omega(u) \in X \text{ with } u \leq t\}$ and let F be the associated Borel σ-algebra to Ω. If $\mathbf{P}$ is a Wiener measure on Ω then $\left(\Omega, F, \{F_t\}_{t\geq 0}, \mathbf{P}\right)$ is a filtered probability space with the Wiener motion $W(t, \omega) = \omega(t)$ for all $(t, \omega) \in R_+ \times \Omega$.

Then $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$ defined by

$$\varphi(t, \omega)(\tau) = \omega(t + \tau) - \omega(t)$$

is a stochastic semiflow on Ω.

Indeed,

$$\begin{aligned}
&\varphi(0, \omega)(s) = \omega(s), \quad \text{and} \\
&\varphi(t + s, \omega)(\tau) = \omega(t + s + \tau) - \omega(t + s) = \varphi(s, \omega)(t + \tau) - \varphi(s, \omega)(t) \\
&\qquad = \varphi(t, \varphi(s, \omega))(\tau).
\end{aligned}$$

Let X be a real separable Hilbert space and let $L(X)$ be the set of all linear bounded operators from X to X.

Definition 2.2. A mapping $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ with the properties

(c1) $\Phi(0, \omega) = I$ (the identity operator on X),
(c2) $\Phi(t + s, \omega) = \Phi(t, \varphi(s, \omega))\Phi(s, \omega)$,
for all $(t, s, \omega) \in R_+^2 \times \Omega$, is called a *stochastic cocycle on* X over the stochastic semiflow $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$.
A stochastic cocycle $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ with the property
(c3) for every $(\omega, x) \in \Omega \times X$ the mapping $t \to E\left(\|\Phi(t, \omega)x\|^2\right)$ is measurable, is called *strongly measurable in mean square*.
A stochastic cocycle $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ with the property
(c4) there are $M \geq 1$ and $\lambda > 0$ such that

$$E\|\Phi(t, \omega)\|^2 \leq M e^{\lambda(t-s)} E\|\Phi(s, \omega)\|^2,$$

for all $t \geq s \geq 0$, and $\omega \in \Omega$, is called a *stochastic cocycle with uniform exponential growth in mean square*.

Example 2.2. Let X be a real separable Hilbert space and let $\{W(t)\}_{t\geq 0}$ be an X-valued Brownian motion with a separable covariance Hilbert space H and defined on the canonical complete filtered Wiener space $\left(\Omega, F, \{F_t\}_{t\geq 0}, \mathbf{P}\right)$ introduced in Example 2.1.

Let $L(H, X)$ be the Banach space of all bounded linear operators from H to X. Denote by $L_2(H, X) \subset L(H, X)$, the Hilbert–Schmidt operators $S : H \to X$, with the norm

$$\|S\| = \left(\sum_{k=1}^{\infty} |S(f_k)|^2\right)^{1/2},$$

where $|.|$ is the norm on H, and $\{f_k, k \geq 1\}$ is a complete orthonormal system on H.

Consider the linear stochastic evolution equation of the form

$$\begin{cases} \mathrm{d}u(t, x, .) = Au(t, x, .)\mathrm{d}t + Bu(t, x, .)\mathrm{d}W(t), & t > 0; \\ u(0, x, \omega) = x \in X, \end{cases}$$

where $A : D(A) \subset X \to X$ is the infinitesimal operator of a strongly continuous semigroup of bounded linear operators $T(t) : X \to X, t \geq 0$, and $B : X \to L_2(H, X)$ is a bounded linear operator. Suppose that B can be extended to a bounded linear operator $B : X \to L(X)$, which can be denoted by the same, and the series $\sum_{k=1}^{\infty} \|B_k^2\|_{L(X)}$ converge, where $B_k : X \to X$ are bounded linear operators defined by $B_k(x) = B(x)(f_k), x \in X, k \geq 1$.

A *mild solution* of this equation is given (see [10,7]) by the family of $\{F_t\}_{t\geq 0}$-adapted processes $u(., x, .) : \mathbf{R}_+ \times \Omega \to X, x \in X$, satisfying the following stochastic integral equation:

$$u(t, x, .) = T(t)x + \int_0^t T(t - \tau)Bu(\tau, x, .)\mathrm{d}W(\tau).$$

Then the mapping $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ defined by

$$\Phi(t, \omega)x = u(t, x, \omega),$$

is a stochastic cocycle over the stochastic semiflow $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$ defined in Example 2.1. (see [10]).

Example 2.3. Let G be the Banach space of X-valued stochastic processes α with the norm

$$\|\alpha\| = \left(\sup_{t\geq 0} E\|\alpha(t)\|^2\right)^{1/2}.$$

If $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ is the stochastic cocycle over the stochastic semiflow $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$ then $\Phi_\alpha : \mathbf{R}_+ \times \Omega \to L(X)$ defined by

$$\Phi_\alpha(t,\omega)x = \Phi(t,\omega)x + \int_0^t \Phi(t-\tau, \varphi(\tau,\omega))\alpha(\tau)\mathrm{d}\tau, \tag{1}$$

is also a stochastic cocycle over the stochastic semiflow φ, which is called an *α-shifted stochastic cocycle of Φ*.

In particular, if Φ is the stochastic cocycle defined in Example 2.2 then Φ_α is the stochastic cocycle associated to the stochastic equation (see [10])

$$\begin{cases} \mathrm{d}u(t,x,.) = (Au(t,x,.) + \alpha(t))\,\mathrm{d}t + Bu(t,x,.)\mathrm{d}W(t), \quad t > 0; \\ u(0,x,\omega) = x \in X. \end{cases} \tag{2}$$

3. Uniform exponential dichotomy in mean square

Let $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ be a stochastic cocycle on the real Hilbert space over the stochastic semiflow $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$.

In what follows, we denote by X_1 the linear subspace of all $x \in X$ with the property

$$E\|\Phi(0,\omega)x\|^2 < \infty,$$

and this is called the stable manifold. Let X_2 be the complementary subspace of X_1, i.e. we have

$$X = X_1 \oplus X_2.$$

Then we denote by P_1 the projection on X_2 (that is, $\ker P_1 = X_2$) and by $P_2 = I - P_1$, the projection on X_1.

Definition 3.1. The stochastic cocycle $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ is said to be

(1) *uniformly dichotomic in mean square* if the constants $N_1, N_2 \geq 1$ exist, such that:

$$E\|\Phi(t,\omega)x\|^2 \leq N_1 E\|\Phi(s,\omega)x\|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, x \in X_1 \tag{3}$$

$$E\|\Phi(s,\omega)x\|^2 \leq N_2 E\|\Phi(t,\omega)x\|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, x \in X_2 \tag{4}$$

(2) *uniformly exponentially dichotomic in mean square* if the constants $N_1, N_2 \geq 1$, $\nu_1, \nu_2 \geq 0$ exist, such that:

$$E\|\Phi(t,\omega)x\|^2 \leq N_1 \mathrm{e}^{-\nu_1(t-s)} E\|\Phi(s,\omega)x\|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, x \in X_1 \tag{5}$$

$$E\|\Phi(s,\omega)x\|^2 \leq N_2 \mathrm{e}^{-\nu_2(t-s)} E\|\Phi(t,\omega)x\|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, x \in X_2. \tag{6}$$

Lemma 3.1. *Let $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ be a stochastic cocycle with uniform exponential growth in mean square, and with the property that, for each stochastic process $\alpha(t) \in G$ the α-shifted stochastic cocycle Φ_α is bounded in mean square.*

Then there is a constant $K > 0$ such that for every $\alpha(t) \in G$ there exists a unique $x \in X_2$ such that

$$E\|\Phi_\alpha(t,\omega)x\|^2 \leq K \sup_{t\geq 0} E\|\alpha(t)\|^2, \tag{7}$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$.

Proof. Suppose that for an arbitrary stochastic process $\alpha(t) \in G$ the α-shifted stochastic cocycle Φ_α satisfies

$$E\|\Phi_\alpha(t,\omega)x\|^2 < \infty,$$

for all $(t,x,\omega) \in \mathbf{R}_+ \times X \times \Omega$. Let $x_k = P_k x$, for all $x \in X$. Then from relation

$$\Phi_\alpha(t,\omega)x = \Phi(t,\omega)x_1 + \Phi_\alpha(t,\omega)x_2,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$, it results that $\Phi_\alpha(t,\omega)x$, $x \in X$ is an α-shifted stochastic cocycle of Φ, and from the fact that $\Phi(t,\omega)x_1$ is a stochastic cocycle bounded in mean square, it results that $\Phi_\alpha(t,\omega)x_2$ is an α-shifted stochastic cocycle with the properties

$$\sup_{t\geq 0} E\|\Phi_\alpha(t,\omega)x_2\|^2 < \infty \quad \text{and}$$
$$\Phi_\alpha(0,\omega)x_2 = x - P_1x = (I-P_1)x = P_2x \in X_2.$$

Thus for all $t \geq 0$ and $\alpha(t) \in G$, there exists $x \in X_2$ such that $\Phi_\alpha(t,\omega)x$ is an α-shifted stochastic cocycle of Φ with the property

$$\sup_{t\geq 0} E\|\Phi_\alpha(t,\omega)x\|^2 < \infty.$$

Assume that, for $\alpha(t) \in G$, $t \geq 0$, there exists $x_2', x_2'' \in X_2$ such that

$$\sup_{t\geq 0} E\|\Phi_\alpha(t,\omega)x_2'\|^2 < \infty, \qquad \sup_{t\geq 0} E\|\Phi_\alpha(t,\omega)x_2''\|^2 < \infty,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. Then we have

$$\begin{aligned}
E\|\Phi_\alpha(t,\omega)x_2' - \Phi_\alpha(t,\omega)x_2''\|^2 &\leq 2E\|\Phi(t,\omega)x_2' - \Phi(t,\omega)x_2''\|^2 \\
&\quad + 2\int_0^t E\|\Phi(t-\tau,\omega)x_2' - \Phi(t-\tau,\omega)x_2''\|^2 E\|\alpha(\tau)\|^2 \mathrm{d}\tau \\
&\leq 2\left(M\mathrm{e}^{\lambda t}E\|x_2'-x_2''\|^2 + NM\int_0^t \mathrm{e}^{\lambda(t-\tau)}\mathrm{d}\tau\, E\|x_2'-x_2''\|^2\right) \\
&\leq 2M\mathrm{e}^{\lambda t}\left(1+\frac{N}{\lambda}\right)E\|x_2'-x_2''\|^2 < \infty
\end{aligned}$$

where M, λ are constants from Definition 3.1, and $N = \sup_{t\geq 0}\|\alpha(t)\|^2$. Thus we obtain

$$x_2' - x_2'' \in X_1 \cap X_2 = 0,$$

and hence $x_2' = x_2''$.

Let us consider the space G_1 of all α-shifted stochastic cocycles of $\Phi, \alpha(t) \in G$, with the condition $\Phi_\alpha(0,\omega)x \in X_2$ bounded in mean square. Let $C : G \to G_1$ be the linear operator defined by

$$C\alpha = \Phi_\alpha(t,\omega)x,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. To establish property (7) it is enough to show by the closed graph theorem that C is a closed operator. Let $\{\alpha_n\}$ be a sequence of G; thus there exists a limit of this in G, such that

$$E\|\alpha_n(t) - \alpha(t)\|^2 \to 0, \quad n \to \infty,\ \forall t \geq 0.$$

Then there exists $y \in G_1$ such that

$$E\|\Phi_{\alpha_n}(t,\omega)x - y(t)\|^2 \to 0, \quad n \to \infty,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. It follows that there exist a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$, and $x_k \in X_2$ such that

$$E\|\Phi_{\alpha_{n_k}}(t,\omega)x_k - y(t)\|^2 \to 0, \quad k \to \infty,$$

in the space G_1, for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. From

$$\begin{aligned}
&E\|y(t) - \Phi_\alpha(t,\omega)x\|^2 \\
&\quad \le 2\left(E\|y(t) - \Phi_{\alpha_{n_k}}(t,\omega)x_k\|^2 + E\|\Phi_{\alpha_{n_k}}(t,\omega)x_k - \Phi_\alpha(t,\omega)x\|^2\right) \\
&\quad \le 2E\|y(t) - \Phi_{\alpha_{n_k}}(t,\omega)x_k\|^2 + 4E\|\Phi(t,\omega)x_k - \Phi(t,\omega)x\|^2 \\
&\qquad + 4\int_0^t E\|\Phi(t-\tau,\omega)x_k - \Phi(t-\tau,\omega)x\|^2 E\|\alpha_{n_k}(\tau) - \alpha(\tau)\|^2 \mathrm{d}\tau \\
&\quad \le 2E\|y(t) - \Phi_{\alpha_{n_k}}(t,\omega)x_k\|^2 + 4M\mathrm{e}^{\lambda t}E\|x_k - x\|^2\left(1 + \frac{1}{\lambda}\sup_{t\ge 0} E\|\alpha_{n_k} - \alpha\|^2\right)
\end{aligned}$$

for $k \to \infty$ we obtain that

$$y(t) = \Phi_\alpha(t,\omega)x,$$

for all $t \ge 0$, with a probability of 1. Hence $\sup_{t\ge 0} E\|\Phi_\alpha(t,\omega)x\|^2 < \infty$, and thus we have

$$C\alpha(t) = \Phi_\alpha(t,\omega)x = y(t), \quad \forall\, t \ge 0. \quad \square$$

The main result of this paper is the next theorem. In the papers [12,13] the authors proves the connections between some "Perron-type" conditions and the exponential dichotomy in deterministic case, and in [6,9] is presents the asymptotic behavior of skew-product semiflows in Banach spaces.

Theorem 3.1. *If the stochastic cocycle $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ satisfies the hypothesis from Lemma 3.1, then it is uniformly exponential dichotomic in mean square.*

Proof. Let $x \in X$ and $\alpha(t) = \frac{\chi(t)}{(E\|\Phi(t,\omega)x\|^2)^{1/2}}\Phi(t,\omega)x$ where

$$\chi(t) = \begin{cases} 1, & t \in [0, t_0+\tau]; \\ 1 - (t - t_0 - \tau), & t \in [t_0+\tau, t_0+\tau+1]; \\ 0, & t \ge t_0 + \tau \end{cases}$$

for all $\omega \in \Omega$. Obviously $\sup_{t\ge 0}\alpha(t) < \infty$, and the α-shifted stochastic cocycle $\Phi_\alpha(t,\omega)x$ of Φ is from G. It results, from Lemma 3.1, that there exists a unique $x \in X_2$ such that

$$\left(E\|\Phi_\alpha(t,\omega)x\|^2\right)^{1/2} \le K\left(\sup_{t\ge 0} E\|\alpha(t)\|^2\right)^{1/2} \le K,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. In particular, if $t = t_0 + \tau$ then

$$\left(E\|\Phi_\alpha(t,\omega)x\|^2\right)^{1/2} = \left(E\left\|\Phi(t_0+\tau,\omega)x + \int_0^{t_0+\tau}\Phi(t_0+\tau-s,\omega)x\alpha(s)\mathrm{d}s\right\|\right)^{1/2}$$

$$\leq \left(2E\|\Phi(t_0+\tau,\omega)x\|^2\left(1+\int_0^{t_0+\tau}\frac{1}{E\|\Phi(s,\omega)x\|^2}\mathrm{d}s\right)\right)^{1/2} \leq K.$$

Thus we have

$$\int_0^{t_0+\tau}\frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s \leq K\frac{1}{\left(E\|\Phi(t_0+\tau,\omega)x\|^2\right)^{1/2}}. \tag{8}$$

Let us consider the function

$$\delta(t) = \int_0^t \frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s.$$

Since the second moment of the stochastic cocycle satisfies a system of ordinary linear differential equations [see [7]], this function is continuously differentiable. Then, from (8), it results that

$$\frac{\delta'(t_0+\tau)}{\delta(t_0+\tau)} \geq \frac{1}{K}.$$

Integrating this inequality on $[1,\tau]$, we have

$$\delta(t_0+\tau) \geq \mathrm{e}^{\frac{1}{K}(\tau-1)}\delta(t_0+1), \quad \forall \tau \geq 1. \tag{9}$$

If $t \in [t_0, t_0+1]$, from uniform exponential growth in mean square of stochastic cocycle Φ we have

$$E\|\Phi(t,\omega)x\|^2 \leq M\mathrm{e}^{\lambda(t-t_0)}E\|\Phi(t_0,\omega)x\|^2, \tag{10}$$

for all $(\omega, x) \in \Omega \times X$. By integration on $[t_0, t_0+1]$, we obtain

$$\delta(t_0+1) = \int_{t_0}^{t_0+1}\frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s \geq \frac{2\left(1-\mathrm{e}^{-\frac{2}{\lambda}}\right)}{\lambda M^{1/2}}\frac{1}{\left(E\|\Phi(t_0,\omega)x\|^2\right)^{1/2}}.$$

For $\tau \geq 1$, and from relation (7) we have

$$\begin{aligned}\left(E\|\Phi(t_0+\tau,\omega)x\|^2\right)^{1/2} &= \frac{1}{\delta'(t_0+\tau)} \leq \frac{K}{\delta(t_0+\tau)} \leq \frac{K}{\delta(t_0+1)}\mathrm{e}^{-\frac{1}{K}(\tau-1)}\\ &\leq \frac{M^{1/2}K}{\frac{2}{\lambda}\left(1-\mathrm{e}^{-\frac{2}{\lambda}}\right)}\mathrm{e}^{-\frac{1}{K}(\tau-1)}\left(E\|\Phi(t_0,\omega)x\|^2\right)^{1/2}.\end{aligned}$$

Denoting $N = \frac{M^{1/2}K\mathrm{e}^{1/K}}{\frac{2}{\lambda}(1-\mathrm{e}^{-2/\lambda})}$ results in

$$\left(E\|\Phi(t_0+\tau,\omega)x\|^2\right)^{1/2} \leq N\mathrm{e}^{-\frac{\tau}{K}}\left(E\|\Phi(t_0,\omega)x\|^2\right)^{1/2}. \tag{11}$$

From relation (10), if $t = t_0+\tau$, $\tau \leq 1$ we have

$$E\|\Phi(t_0+\tau,\omega)x\|^2 \leq M\mathrm{e}^{\lambda\tau}E\|\Phi(t_0,\omega)x\|^2, \tag{12}$$

for all $(\omega, x) \in \Omega \times X$. For $t = t_0+\tau$, from relations (11) and (12) we obtain

$$E\|\Phi(t,\omega)x\|^2 \leq N_1\mathrm{e}^{-\nu_1(t-t_0)}E\|\Phi(t_0,\omega)x\|^2, \quad \forall t \geq t_0 \geq 0,\ \forall x \in X_1 \tag{13}$$

where $N_1 = \max(N^2, M)$, and $\nu_1 = 1/K$.

Let $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ be a stochastic cocycle with uniformly exponential growth in mean square and $\Phi(0, \omega)x = x \in X_2$. If we take the function $\alpha(t) = -\frac{\chi(t)}{\left(E\|\Phi(t,\omega)x\|^2\right)^{1/2}} \Phi(t, \omega)x$, it results that $\alpha(t)$ is from G. Thus we have

$$\int_0^t \Phi(t-\tau,\omega)x\alpha(\tau)\mathrm{d}\tau = -\int_0^t \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}} \Phi(t-\tau,\omega)\Phi(\tau,\omega)x\mathrm{d}\tau$$
$$= -\Phi(t,\omega)x \int_0^t \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}}\mathrm{d}\tau$$
$$= \Phi(t,\omega)x \int_t^\infty \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}}\mathrm{d}\tau - \Phi(t,\omega)x \int_0^\infty \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}}\mathrm{d}\tau.$$

Passing to the norm in G, and using the uniform exponential growth in mean square of stochastic cocycle Φ, we have that $\int_0^t E\|\Phi(t-\tau,\omega)x\|^2 E\|\alpha(\tau)\|^2 \mathrm{d}\tau < \infty$. Thus for this α, from Lemma 3.1, it results that there exists a unique $x \in X_2$ such that

$$\left(E\|\Phi_\alpha(t,\omega)x\|^2\right)^{1/2} \le \left(E\left\|\Phi(t,\omega)x\int_t^\infty \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}}\mathrm{d}\tau\right\|\right)^{1/2} \le K,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. Then, for all $\tau \ge 0$ and $t \ge 0$, we have

$$\int_t^\infty \frac{\chi(\tau)}{\left(E\|\Phi(\tau,\omega)x\|^2\right)^{1/2}}\mathrm{d}\tau \le \frac{K}{(E\|\Phi(t,\omega)x\|)^{1/2}}.$$

Because the integral of the left side is monotone increasing for $\tau \ge 0$ and is bounded, it results that it has a limit for $\tau \to \infty$, but

$$\int_t^\infty \frac{\chi(s)}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s = \int_t^{t_0+\tau} \frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s + \int_{t_0+\tau}^{t_0+\tau+1} \frac{\chi(\tau)}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s.$$

Since the second integral is convergent, it tends to zero for $\tau \to \infty$ and we get

$$\int_t^\infty \frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s \le \frac{K}{(E\|\Phi(t,\omega)x\|)^{1/2}}. \tag{14}$$

If we consider the function

$$\delta(t) = \int_t^\infty \frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s,$$

then it follows that

$$\delta'(t) \le -\frac{1}{K}\delta(t).$$

Integrating this inequality on $[t_0, t]$ results in

$$\delta(t) \le \mathrm{e}^{-\frac{1}{K}(t-t_0)}\delta(t_0). \tag{15}$$

Since the stochastic cocycle Φ has uniform exponential growth in mean square, there exist the positive constants M, λ such that

$$E\|\Phi(\tau,\omega)x\|^2 \le M\mathrm{e}^{\lambda(\tau-t)}E\|\Phi(t,\omega)x\|^2, \tag{16}$$

for all $(\omega, x) \in \Omega \times X$ and $\tau \ge t$. Thus we obtain

$$\begin{aligned}\delta(t)\left(E\|\Phi(t,\omega)x\|^2\right)^{1/2} &= \left(E\|\Phi(t,\omega)x\|^2\right)^{1/2}\int_t^\infty \frac{1}{\left(E\|\Phi(s,\omega)x\|^2\right)^{1/2}}\mathrm{d}s\\ &\ge \int_t^\infty \frac{1}{M}\mathrm{e}^{-\lambda(s-t)}\mathrm{d}s = L,\end{aligned}$$

where L is a positive constant. From the relations (14) and (15) it results that

$$\left(E\|\Phi(t,\omega)x\|^2\right)^{1/2} \ge \frac{L}{\delta(t)} \ge \frac{L}{\delta(t_0)}\mathrm{e}^{\frac{1}{K}(t-t_0)} \ge \frac{L}{K}\mathrm{e}^{\frac{1}{K}(t-t_0)}\left(E\|\Phi(t_0,\omega)x\|^2\right)^{1/2}.$$

Denoting $K_2 = \frac{L^2}{K^2}$, $\nu_2 = \frac{2}{K}$, we have

$$E\|\Phi(s,\omega)x\|^2 \le N_2\mathrm{e}^{-\nu_2(t-s)}E\|\Phi(t,\omega)x\|^2, \quad \forall\, t \ge s \ge 0,\ \forall\, x \in X_2. \tag{17}$$

It results from inequalities (13) and (17) that, the stochastic cocycle $\Phi : \mathbf{R}_+ \times \Omega \to L(X)$ is uniformly exponentially dichotomic in mean square. □

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